Finite resolution of time in continuous measurements: phenomenology and the model*

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Abstract

Definition of a quantum corridor describing monitoring a quantum observable in the framework of the phenomenological restricted-path-integral (RPI) approach is generalized for the case of a finite resolution of time. The resulting evolution of the continuously measured system cannot be presented by a differential equation. Monitoring the position of a quantum particle is also considered with the help of a model which takes into account a finite resolution of time. The results based on the model are shown to coincide with those of the phenomenological approach.

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1 Introduction

During last decades the theory of continuous quantum measurements has been under thorough investigation both with the help of models [1, 2, 3, 4] and in the framework of different phenomenological approaches [5, 6, 7, 8] The interest to this field significantly increased in connection with the quantum

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Zeno effect predicted in [9] and experimentally verified in [10]. Phenomenological approaches have an advantage of being model-independent. One of the first approaches of this type applied to continuous quantum measurements was one based on restricted path integrals (RPI). It was proposed by the author [6, 7, 8] (see also [11]) following an idea of R.Feynman [12]. A Lindblad-type master equation for the density matrix of an open measured system can be derived from this approach [13]. Analogous equations follow from concrete models of continuous quantum measurements [2, 3, 4].

To describe monitoring a quantum observable in the framework of the RPI approach, one has to define *quantum corridors*, corresponding to different readouts of the measurement. In the preceding works the quantum corridors were used which corresponded to the assumption that monitoring is performed with the ideal resolution of time. In the present paper we shall consider a more general definition of quantum corridors including the effect of a *finite resolution of time*. The evolution of a measured system is presented in the resulting theory by an influence functional, but it cannot be described by a differential equation (for example a master equation).

Finally the results of the phenomenological consideration will be compared with conclusions based on a model. For this goal a modification of the model [4] will be presented which allows one to take into account a finite resolution of time. The description of the measured system following from the modified model will be shown to agree with the conclusions of the RPI approach.

2 Quantum corridors

A measured system is considered in the RPI theory of continuous measurements as an open system. The back influence of a measuring device (environment) onto the measured system is taken into account by restricting the Feynman path integral presenting the propagator. The restriction is determined by the information about the measured system supplied by the measurement. Let us outline this approach (see [8] for details).

The evolution of a *closed* quantum system during a time interval T is described by the evolution operator U_T. A matrix element of this operator between the states with definite positions (in the configuration space) is called the *propagator* and may be expressed in the form of the Feynman path

integral (the variables q and p may be multidimensional)

$$U_T(q'', q') = \langle q''|U_T|q'\rangle = \int_{q'}^{q''} \mathrm{d}[p,q]\, \exp\left[\frac{i}{\hbar}\int_0^T (p\dot{q} - H(p,q,t))\right]. \tag{1}$$

If the system undergoes a continuous (prolonged in time) measurement and therefore is considered as being *open*, its evolution may be described (in the RPI approach) by the set of *partial evolution operators* U_T^α depending on outputs (readouts) α of the measurement:

$$|\psi_T^\alpha\rangle = U_T^\alpha|\psi_0\rangle, \quad \rho_T^\alpha = U_T^\alpha \rho_0\, (U_T^\alpha)^\dagger\,. \tag{2}$$

The *partial propagators* are expressed by restricted path integrals. This means that the path integral for U_T^α must be of the form (1) but with the integration restricted according to the information given by the measurement readout α. The information given by α may be presented by a weight functional $w_\alpha[p,q]$ (positive, with values between 0 and 1) so that the partial propagator has to be written as a weighted path integral

$$U_T^\alpha(q'', q') = \langle q''|U_T^\alpha|q'\rangle = \int_{q'}^{q''} \mathrm{d}[p,q]\, w_\alpha[p,q]\, \exp\left[\frac{i}{\hbar}\int_0^T (p\dot{q} - H(p,q,t))\right]. \tag{3}$$

The probability density for α to arise as a measurement readout is given by the trace of the density matrix ρ_T^α so that the probability for α to belong to some set $\mathcal{A}$ of readouts is

$$\mathrm{Prob}(\alpha \in \mathcal{A}) = \int_{\mathcal{A}} \mathrm{d}\alpha\, \mathrm{Tr}\, \rho_T^\alpha \tag{4}$$

with an appropriate measure $\mathrm{d}\alpha$ on the set of readouts.

The preceding consideration concerns the situation when the measurement readout α is known (a *selective* description of the measurement). If the readout is unknown (a *non-selective* description), the evolution of the measured system may be presented by the complete density matrix

$$\rho_T = \int \mathrm{d}\alpha\, \rho_T^\alpha = \int \mathrm{d}\alpha\, U_T^\alpha \rho_0\, (U_T^\alpha)^\dagger\,. \tag{5}$$

The generalized unitarity condition

$$\int \mathrm{d}\alpha\ (U_T^\alpha)^\dagger\ U_T^\alpha = \mathbf{1} \tag{6}$$

provides conservation of probabilities.

In the special case, when *monitoring an observable* $A = A(p, q, t)$ is considered as a continuous measurement, the measurement readout is given by the curve

$$[a] = \{a(t) | 0 \leq t \leq T\}$$

characterizing values of this observable in different time moments. If the square average deflection is taken as a measure of the deviation of the observable $A(t) = A(p(t), q(t), t)$ from the readout $a(t)$, then the weight functional describing the measurement may be taken[1] in the Gaussian form:

$$w_{[a]}[p, q] = \exp\left[-\kappa \int_0^T (A(t) - a(t))^2 \,\mathrm{d}t\right]. \tag{7}$$

The constant κ characterizes the resolution of the measurement and may be expressed in terms of the "measurement error" Δa_T achieved during the period T of the measurement, $\kappa = 1/T\Delta a_T^2$. The error Δa_T decreases with the duration T of the measurement increased, $\Delta a_T \sim 1/\sqrt{T}$.

The resulting path integral

$$U_T^{[a]}(q'', q') = \int_{q'}^{q''} \mathrm{d}[p, q] \exp\left\{\frac{i}{\hbar}\int_0^T (p\dot{q} - H)\mathrm{d}t - \kappa \int_0^T \left(A(t) - a(t)\right)^2 \mathrm{d}t\right\} \tag{8}$$

has the form of a conventional (non-restricted) Feynman path integral (1) but with the non-Hermitian *effective Hamiltonian*

$$H_{[a]}(p, q, t) = H(p, q, t) - i\kappa\hbar \left(A(p, q, t) - a(t)\right)^2 \tag{9}$$

instead of the original Hamiltonian H. Therefore, the partial propagator (8) satisfies a Schrödinger equation with the effective Hamiltonian.

This allows one to describe a continuous measurement (monitoring) without calculating a restricted path integral. Instead, one may solve the Schrödinger equation (with the effective Hamiltonian) for a wave function of the system:

$$\frac{\partial}{\partial t}|\psi_t\rangle = -\frac{i}{\hbar}H_{[a]} = \left(-\frac{i}{\hbar}H - \kappa\left(A - a(t)\right)^2\right)|\psi_t\rangle. \tag{10}$$

[1] The choice of the weight functional depends on the class of measurements under consideration.

If the initial wave function ψ_0 corresponds to the initial state of the measured system, then the solution ψ_T in the final time moment presents the state of the system after the measurement, under the condition that the measurement readout is $[a]$.

The wave function ψ_T obtained in this way has a non-unit norm. If the initial wave function is normalized, then the norm of the final wave function, according to Eq. (4), determines the probability density of the measurement output: $P[a] = ||\psi_T||^2$. Solving the Schrödinger equation for the same initial condition but for different readouts $[a]$, one has a probability distribution over all possible scenarios of the measurement with the corresponding final states of the measured system.

The *non-selective description* of the measurement (if the readout is unknown) is given by the density matrix ρ_t defined by (5) and satisfying [13] the equation

$$\dot{\rho} = -\frac{i}{\hbar}[H, \rho] - \frac{\kappa}{2}[A, [A, \rho]]. \tag{11}$$

The influence of the measuring device (measuring medium) on the measured system may be described by an *influence functional* $W[p, q|p', q']$ (see [14]) in the sense that the 'superpropagator' describing the evolution of the density matrix

$$\rho_T(q, q') = \int \mathrm{d}q_0 \mathrm{d}q_0' \, U(q, q'|q_0, q_0') \rho_0(q_0, q_0') \tag{12}$$

may be presented in the form of a double path integral:

$$\begin{aligned} U(q, q'|q_0, q_0') &= \int_q^{q_0} \mathrm{d}[p, q] \int_{q'}^{q_0'} \mathrm{d}[p', q'] \, W[p, q|p', q'] \\ \times \quad & \exp\left[\frac{i}{\hbar}\int_0^T [(p\dot{q} - H(p, q, t)) - (p'\dot{q}' - H(p', q', t))]\right] \end{aligned} \tag{13}$$

In the present case the influence functional may easily be derived from Eq. (5) and has the form

$$W[p, q|p', q'] = \int \mathrm{d}[a] \, w_{[a]}[p, q] \, w_{[a]}[p', q']. \tag{14}$$

With the weight functional (7) this gives, up to an inessential number factor,

$$W[p, q|p', q'] = \exp\left[-\frac{\kappa}{2}\int_0^T \mathrm{d}t \, (A(p, q, t) - A(p', q', t))^2\right]. \tag{15}$$

3 Quantum corridors for a finite time resolution

It has been assumed in the preceding arguments that the time is measured precisely. This means that the number $a(t)$ is an estimate, supplied by the measurement, of a value $A(t)$ of the observable A at the precisely known instant t. This assumption is not always realistic. A real device gives an estimate of the observable A over some finite time interval of the duration, say, τ. We shall refer to this situation as "measuring time with the resolution τ".

To be more concrete, let $a(t)$ be an estimate, due to the measurement, of the entity

$$\overline{A}(t) = \langle A \rangle_t = \int \mathrm{d}t' \, \Pi_t(t') \, A(t') \tag{16}$$

defined by an appropriate 'form-factor' Π_t (depending on the type of the measuring device). Then, instead of Eq. (7), the weight functional for restricting the path integral should be defined as follows:

$$\tilde{w}_{[a]}[p, q] = \exp\left[-\kappa \int_0^T (a(t) - \overline{A}(t))^2 \, \mathrm{d}t \right]. \tag{17}$$

We arrive therefore, instead of Eq. (8), to the following expression for the partial propagator:

$$U_T^{[a]}(q'', q') = \int_{q'}^{q''} \mathrm{d}[p, q] \exp\left\{ \frac{i}{\hbar} \int_0^T (p\dot{q} - H)\mathrm{d}t - \kappa \int_0^T (\overline{A}(t) - a(t))^2 \mathrm{d}t \right\}. \tag{18}$$

It differs radically in that $\overline{A}(t)$ depends on $A(t')$ for different time moments t'. Therefore, the description of the evolution is *'not local in time'* and cannot be reduced to an effective Hamiltonian in analogy with Eqs. (9, 10).

The partial propagators (18) describe an evolution of the measured system selectively, i.e. with the measurement output $[a]$ taken into account. A non-selective description is given by the general formula (5) resulting in the present case in an influence functional of the form

$$\begin{aligned} \tilde{W}[p, q | p', q'] &= \int \mathrm{d}[a] \, \tilde{w}_{[a]}[p, q] \, \tilde{w}_{[a]}[p', q'] \\ &= \exp\left[-\frac{\kappa}{2} \int_0^T \mathrm{d}t \, (\overline{A}(p, q, t) - \overline{A}(p', q', t))^2 \right]. \end{aligned} \tag{19}$$

Remark 1 *The measure* $\mathrm{d}[a]$ *of integration over measurement readouts has to be chosen in such a way that the generalized unitarity Eq. (6) be valid. Eq. (19) is valid with the conventional functional measure* $\mathrm{d}[a] \sim \prod_t \mathrm{d}a(t)$ *which provides the generalized unitarity either for a linear measured system or for a system with a not too large nonlinearity and a measurement with a not too high resolution. I a general case a weight has to be included in the measure* $\mathrm{d}[a]$*. Eq. (19) should be modified in this case.*

The influence functional (19) enables one to describe the evolution of the measured system by Eqs. (12, 13) (with $\tilde{W}$ instead of W), but no differential equation in time (analogous to Eq. (11)) exists for the resulting density matrix. This is a consequence of non-locality in time.

In the rest of the paper we shall consider a concrete model of the monitoring the position of a particle to verify that it actually leads to the influence functional of the form Eq. (19).

4 Finite resolution in the model of a continuous measurement

The well-known model of a quantum diffusion proposed by Caldeira and Leggett [2] may be considered as a model for a continuous measurement, namely, for monitoring the position of a particle by a measuring medium. In this model the decoherence (measurement) is caused by the interaction of the particle with modes of the crystal.

One more model of this type has been proposed in [4]. This model also consists of a particle in some medium. However the "atoms" of the medium are modelled as oscillators not interacting with each other. Decoherence is caused in this case by interaction of the particle with internal degrees of freedom of the atoms (presented as degrees of freedom of the oscillators). Let us remark that in most cases such a model is more realistic than the Caldeira-Leggett model because the decoherence due to internal structure of atoms is more efficient (more fast) than the decoherence by modes of a crystal. Now we shall modify this model to take into account a finite resolution of time.

The measuring medium in the model [4] consists of atoms in the nods of a cubic lattice. Each atom is presented as an oscillator. We shall present

each atom as a family of oscillators of different frequencies. However let us begin by the case of a single oscillator with the frequency ω as in [4]. The Hamiltonian of the interaction between the oscillator and the particle is taken in [4] as

$$H_{\rm int} = \sum_k H_k = \sum_k \gamma_\omega q_k \exp\left[-\frac{(\mathbf{r}-\mathbf{c}_k)^2}{l^2}\right]. \tag{20}$$

Here $\mathbf{r}$ is a position of the measured particle while q_k is a canonical coordinate of the kth oscillator, $\mathbf{c}_k$ its location, γ_ω the interaction constant, and the length l characterizes the range of the interaction. The interaction of the particle with each of the atoms (oscillators) may be considered as a measurement of its location with the precision l.

To solve this model, the discrete lattice of atoms was replaced in [4] by a continuous distribution of them with the constant density n. This enables one to calculate the influence functional describing the influence of the medium on the particle:

$$\begin{aligned} W[\mathbf{r}_1,\mathbf{r}_2] \;&=\; \exp\left\{-\int_0^T {\rm d}t\int_0^T {\rm d}t'\;\nu(\omega)\cos\omega(t-t')\left[\exp\left(-\frac{(\mathbf{r}_1(t)-\mathbf{r}_1(t'))^2}{2l^2}\right)\right.\right.\\ &+\;\left.\left.\exp\left(-\frac{(\mathbf{r}_2(t)-\mathbf{r}_2(t'))^2}{2l^2}\right)-2\exp\left(-\frac{(\mathbf{r}_2(t)-\mathbf{r}_1(t'))^2}{2l^2}\right)\right]\right\} \end{aligned} \tag{21}$$

where

$$\nu(\omega) = n\left(\frac{\pi l^2}{2}\right)^{3/2}\frac{\gamma_\omega^2}{4\hbar m\omega}. \tag{22}$$

This simple model is not enough realistic because it does not describe measuring of time: the state of an oscillator after the interaction with the particle contains no information about the time of the interaction. To describe the measurement of time, it was assumed in [4] that the interaction of each oscillator is turned on for a short time. Let us consider now a more realistic model for the measurement of time.

For this goal, assume that each atom has a more complicated internal structure so that the information about the time of the interaction is recorded in its state. To give a model of the internal structure of the atom, let us present it as a family of oscillators with different frequencies. Now the information about the time of interaction with the particle is recorded in phase relations between different oscillators of the same 'atom'.

Going over to the calculation, we have now integrate the exponent in Eq. (21) over the frequencies ω of the oscillators forming the model of an atom. The influence functional takes the form

$$W[\mathbf{r}_1,\mathbf{r}_2] = \exp\left\{-\frac{\kappa l^2}{2}\int_0^T \mathrm{d}t\int_0^T \mathrm{d}t'\;\Pi(t-t')\left[\exp\left(-\frac{(\mathbf{r}_1(t)-\mathbf{r}_1(t'))^2}{2l^2}\right)\right.\right.$$
$$\left.\left.+\exp\left(-\frac{(\mathbf{r}_2(t)-\mathbf{r}_2(t'))^2}{2l^2}\right)-2\exp\left(-\frac{(\mathbf{r}_2(t)-\mathbf{r}_1(t'))^2}{2l^2}\right)\right]\;\right\} \tag{23}$$

with

$$\Pi(t)=\frac{2}{\kappa l^2}\int \nu(\omega)\cos\omega t\mathrm{d}\omega, \qquad \int\Pi(t)\,\mathrm{d}t = 1. \tag{24}$$

Let the width of the 'form-factor' $\Pi(t)$ be of the order of τ. Then for not too large initial energy of the particle, $E_0 \ll Ml^2/2\tau^2$, the exponentials in the square brackets may be evaluated up to the first order to give

$$W[\mathbf{r}_1,\mathbf{r}_2] = \exp\left\{-\frac{\kappa}{4}\int_0^T \mathrm{d}t'\int_0^T \mathrm{d}t''\;\Pi(t'-t'')\right.$$
$$\left.[2(\mathbf{r}_2(t')-\mathbf{r}_1(t''))^2-(\mathbf{r}_1(t')-\mathbf{r}_1(t''))^2-(\mathbf{r}_2(t')-\mathbf{r}_2(t''))^2]\;\right\} \tag{25}$$

Let the function $\Pi(t)$ be symmetrical. Then it may be expressed through the function of two arguments $\Pi_t(t')$ (depending only on the difference $|t-t'|$):

$$\Pi(t'-t'') = \int \mathrm{d}t\,\Pi_t(t')\,\Pi_t(t''), \qquad \int\Pi_t(t')\mathrm{d}t' = 1. \tag{26}$$

Let us substitute this expression in Eq. (25) and make use of the relation

$$\begin{aligned}&\int\mathrm{d}t'\int\mathrm{d}t''\,\Pi_t(t')\,\Pi_t(t'')\\ &\times\;\; [2(\mathbf{r}_2(t')-\mathbf{r}_1(t''))^2-(\mathbf{r}_1(t')-\mathbf{r}_1(t''))^2-(\mathbf{r}_2(t')-\mathbf{r}_2(t''))^2]\\ &=\;\; 2\int\mathrm{d}t[\langle\mathbf{r}_2\rangle_t-\langle\mathbf{r}_1\rangle_t]^2\end{aligned} \tag{27}$$

where the notation (16) is exploited. Then we immediately see that the influence functional has the form

$$W[\mathbf{r},\mathbf{r}'] = \exp\left[-\frac{\kappa}{2}\int_0^T\mathrm{d}t\,(\langle\mathbf{r}\rangle_t-\langle\mathbf{r}'\rangle_t)^2\right] \tag{28}$$

in accord with Eq. (19). Thus, the prediction of the phenomenological RPI approach coincides with what follows from the concrete model of the measurement even in the case when a finite resolution of time is taken into account.

5 Conclusion

We have shown, both in the RPI phenomenological approach and in the framework of the concrete model of a continuous measurement, that the finiteness of the resolution in measuring time moments leads to the replacement of the value of an observable $A(t)$ by its 'time-coarse-graining' $\overline{A}(t) = \langle A \rangle_t$ in the expression for the influence functional. In a special case when the position $\mathbf{r}(t)$ of a particle is continuously measured, $\mathbf{r}(t)$ has to be replaced by $\langle \mathbf{r} \rangle_t$. As a result, the time evolution of the measured system cannot be presented by a differential equation in time. Neither the master equation Eq. (11) nor the Schrödinger equation with a complex Hamiltonian Eq. (10) is correct in this case.

Time coarse-graining is a characteristic of the 'measuring device' or 'measuring medium'. It characterizes inertial properties of the measuring setup. It is evident that the effect of time coarse-graining is negligible if all physically relevant frequencies are lower than the inverse period of coarse-graining: $\Omega \ll \tau^{-1}$. In this case the master equation and the Schrödinger equation with a complex Hamiltonian are correct. Physically this means that inertial properties of the measuring medium are negligible.

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